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LETTER TO THE EDITOR

The geometric phase on Kähler manifolds

Adrian Stanley

The Physics Laboratory, University of Kent at Canterbury, CT2 7NZ, UK

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Abstract. A new expression for the phase one-form is derived in terms of a derivative of the Kähler potential for the class of projective Hilbert spaces which are Kähler manifolds. The technique used is that of imaginary time-translation, previously introduced by the author.

In a previous paper [1] the author introduced the idea of translating a quantum state in imaginary time as a means of extracting information concerning the geometric phase [2]. As is well known, the geometric phase is a consequence of parallel transport on a projective Hilbert space [3] with a non-trivial global topology [4]. The mapping from the Hilbert space to the projective Hilbert space may be performed in a number of ways. In [1] this was achieved with the aid of projection operators which have the property that they are invariant under multiplication of the state by a phase:

$$|\alpha\rangle \mapsto |\alpha\rangle\langle\alpha| \tag{1}$$

$$e^{i\beta}|\alpha\rangle \mapsto e^{i\beta}|\alpha\rangle\langle\alpha|e^{-i\beta} = |\alpha\rangle\langle\alpha|. \tag{2}$$

In this letter an alternative parametrization in terms of inhomogeneous coordinates will be used. After a brief account of the salient features of Kähler manifolds (of which $\mathcal{CP}^n$ is an example) imaginary time translations will be employed to give a new expression for the phase one-form. A few examples will then be given.

Any non-dissipative quantum state in Hilbert space may be expressed by $n+1$ complex numbers. However, the $n+1$ homogenous coordinates, $Z^0, \cdots, Z^n$ do not form a good coordinate system on the projective Hilbert space, $\mathcal{CP}^n$. In order to define a good set of coordinates we divide through by Z^0 to obtain the inhomogeneous coordinates of the projective Hilbert space:

$$\begin{pmatrix} Z^0 \\ Z^1 \\ \vdots \\ Z^n \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ w^1 \\ \vdots \\ w^n \end{pmatrix} = \begin{pmatrix} \frac{Z^0}{Z^0} \\ \frac{Z^1}{Z^0} \\ \vdots \\ \frac{Z^n}{Z^0} \end{pmatrix}. \tag{3}$$

The existence of a real (1, 1) tensor field called an almost complex structure which performs the following mapping on the tangent space of M, $J_pM : T_pM \mapsto T_pM$,

$$J_p \cdot \frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k} \qquad J_p \cdot \frac{\partial}{\partial y^k} = -\frac{\partial}{\partial x^k} \tag{4}$$

(where $w^k = x^k + \mathrm{i}y^k$) follows from the existence of holomorphic coordinates. Any complex manifold also admits a Hermitian metric defined by

$$g_p(J_pX, J_pY) = g_p(X, Y). \tag{5}$$

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If a tensor field $\Omega(X, Y)$, on a Hermitian manifold is now defined by

$$\Omega_p(X, Y) = g_p(J_p X, Y) \tag{6}$$

then Ω is found to be antisymmetric in X and Y and represents a two-form called the Kähler form of the Hermitian metric. A Kähler manifold is a Hermitian manifold (M, g) with a closed Kähler form, i.e. $d\Omega = 0$. The main point of interest for this letter is that the components of the metric may be written locally as

$$g_{i\bar{j}} = g_{\bar{j}i} = \frac{\partial^2 K}{\partial w^i \partial \bar{w}^j} \tag{7}$$

where K is the Kähler potential of the Kähler metric. A full account of Kähler manifolds and complex manifolds in general may be found in chapter 8 of [5].

If a state vector is chosen whose parametrization over a particular coordinate patch is (Einstein summation convention assumed) [6]:

$$Z^\alpha = \frac{e^{i\phi}}{(1 + \bar{w}_k w^k)^{1/2}} \qquad Z^j = Z^\alpha w^j \tag{8}$$

then the state will be normalised to unity. The metric for such a projective Hilbert space is the well known Fubini–Study metric on $\mathcal{CP}^n$:

$$ds^2(\mathcal{CP}^n) = \frac{(1 + \bar{w}_k w^k)\delta_{ij} - \bar{w}_i w_j}{(1 + \bar{w}_k w^k)^2} dw^i d\bar{w}^j \tag{9}$$

and the associated Kähler potential of the metric is

$$K = \tfrac{1}{2} \ln(1 + \bar{w}_k w^k). \tag{10}$$

If the state is time dependent, then the effect of an infinitesimal imaginary time translation on the Kähler potential may be observed,

$$t \to t - i\epsilon \Rightarrow w^j \to w^j - i\epsilon \dot{w}^j \tag{11}$$

and

$$K(t) \to K(t, \epsilon) = \tfrac{1}{2} \ln(1 + \bar{w}_k w^k - i\epsilon(\bar{w}_k \dot{w}^k - \dot{\bar{w}}_k w^k)) \tag{12}$$

$$= \tfrac{1}{2} \ln(1 + \bar{w}_k w^k + 2\epsilon\, \mathrm{Im}(\bar{w}_k \dot{w}^k)). \tag{13}$$

From the definition of w^k and from the identity $\bar{w}^k \equiv (\bar{Z}^k/\bar{Z}^\alpha)$, it follows that:

$$\bar{w}_k \dot{w}^k = \frac{\bar{Z}_k}{\bar{Z}_\alpha}\left(\frac{\dot{Z}^k}{Z^\alpha} - \frac{Z_k \dot{Z}^\alpha}{Z^{\alpha 2}}\right) \tag{14}$$

$$= \frac{1}{\bar{Z}_\alpha Z^\alpha}\left(\bar{Z}_k \dot{Z}^k - \bar{Z}_k Z^k \frac{\dot{Z}_\alpha}{Z_\alpha}\right) \tag{15}$$

$$= \frac{1}{\bar{Z}_\alpha Z^\alpha}\left(\bar{Z}_k \dot{Z}^k - (1 - \bar{Z}_\alpha Z^\alpha)\frac{\dot{Z}_\alpha}{Z_\alpha}\right) \tag{16}$$

$$= \frac{1}{\bar{Z}_\alpha Z^\alpha}\left(\bar{Z}_k \dot{Z}^k + \bar{Z}_\alpha \dot{Z}^\alpha - \frac{\dot{Z}_\alpha}{Z_\alpha}\right) \tag{17}$$

$$= e^{2K}\left(\bar{Z}_k \dot{Z}^k + \bar{Z}_\alpha \dot{Z}^\alpha - \frac{\dot{Z}_\alpha}{Z_\alpha}\right) \tag{18}$$

where the last equality holds by virtue of

$$e^{2K(t)} = 1 + \bar{w}_k w^k = \frac{\bar{Z}_\alpha Z^\alpha + \bar{Z}_k Z^k}{\bar{Z}_\alpha Z^\alpha} = \frac{1}{\bar{Z}_\alpha Z^\alpha}. \tag{19}$$

Therefore,

$$e^{2K(t,\epsilon)} = 1 + \tilde{w}_k w^k + 2\epsilon\,\mathrm{Im}(\tilde{w}_k \dot{w}^k) \tag{20}$$

$$= e^{2K(t)}(1 + 2\epsilon\,\mathrm{Im}(\bar{Z}_k \dot{Z}^k + \bar{Z}_\alpha \dot{Z}^\alpha - \frac{\dot{Z}_\alpha}{Z_\alpha})) \tag{21}$$

but the phase one-form is defined by

$$\Gamma \equiv \mathrm{Im}(\bar{Z}_k \dot{Z}^k + \bar{Z}_\alpha \dot{Z}^\alpha) \tag{22}$$

whence it follows that

$$K(t,\epsilon) = K(t) + \frac{1}{2}\ln(1 + 2\epsilon(\Gamma - \mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}t}\ln Z_\alpha)) \tag{23}$$

and therefore, over a patch where $Z_\alpha \neq 0$,

$$\left.\frac{\partial K}{\partial \epsilon}\right|_{\epsilon=0} = \Gamma - \mathrm{Im}\frac{\mathrm{d}}{\mathrm{d}t}\ln Z_\alpha. \tag{24}$$

Now, it is clear that the second term on the right-hand side is nothing more than $\dot{\phi}$ and that the phase γ is given by

$$\gamma = \int_0^T \left.\frac{\partial K}{\partial \epsilon}\right|_{\epsilon=0} \mathrm{d}t - [\phi]_0^T. \tag{25}$$

The phase ϕ represents a global phase and is determined by the parallel transport condition on the phase of the state. This is conventionally taken as $\dot{\phi} = 0$ [7]. Therefore, over a coordinate patch where $Z_\alpha \neq 0$ we have

$$\gamma = \int_0^T \left.\frac{\partial K}{\partial \epsilon}\right|_{\epsilon=0} \mathrm{d}t. \tag{26}$$

From the above derivation it is clear that γ is independent of the particular choice of Z_α. The Kähler potential is unique up to the addition of a pluriharmonic function f (defined by $\partial^2 f/(\partial w^i \partial \bar{w})^j = 0$). Such a function will be a sum of terms of the form $f(w^i)$ or $g(\bar{w}^j)$ and the effect of an imaginary time translation on these expressions is

$$f(w^i) \to f(w^i - \mathrm{i}\epsilon\dot{w}^i) = f(w^i) - \mathrm{i}\epsilon\dot{w}^i\frac{\partial f}{\partial w^i} \tag{27}$$

$$= f(w^i) - \mathrm{i}\epsilon\frac{\mathrm{d}f}{\mathrm{d}t}. \tag{28}$$

The first-order term in ϵ gives the connection and here it is $\mathrm{d}f/\mathrm{d}t$, a total time derivative. There will therefore be no contribution to the phase if a pluriharmonic function is added to the Kähler potential.

At this point it is worth a moment's digression to make a few remarks on imaginary time translation as a mechanism. In order to specify a connection on a fibre bundle, a horizontal subspace must be allocated to each point of the tangent space. If we make the following standard decomposition,

$$|\psi\rangle = e^{-\mathrm{i}\alpha}|\tilde{\psi}\rangle \tag{29}$$

(where $|\psi\rangle$ is in Hilbert space (total space), $e^{-\mathrm{i}\alpha}$ is the global phase (fibre) and $|\tilde{\psi}\rangle$ is in projective Hilbert space (base space)) then the projection

$$\langle\psi|\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = 0 \tag{30}$$

defines the usual connection. Let us now make a few comparisons between the derivative $\mathrm{d}/\mathrm{d}t\,(\mathrm{e}^{-\mathrm{i}\alpha}|\tilde{\psi}\rangle)$ and the imaginary time translated state $|\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\mathrm{i}\alpha}|\tilde{\psi}\rangle) = \mathrm{e}^{-\mathrm{i}\alpha}(|\dot{\tilde{\psi}}\rangle - \mathrm{i}\dot{\alpha}|\tilde{\psi}\rangle) \tag{31}$$

whereas

$$|\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle = |\tilde{\psi}\rangle - \mathrm{i}\epsilon|\dot{\tilde{\psi}}\rangle. \tag{32}$$

If we also compare projections onto the base space, namely

$$\langle\tilde{\psi}|\mathrm{e}^{\mathrm{i}\alpha}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\mathrm{i}\alpha}|\tilde{\psi}\rangle) = -\mathrm{i}\dot{\alpha} + \langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle \tag{33}$$

and

$$\langle\tilde{\psi}(t)|\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle = 1 - \mathrm{i}\epsilon\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle \tag{34}$$

then the imaginary part of (33) gives the connection condition on the phase which is conveniently expressed either in terms of the imaginary derivative of (34) or the main result of [1]:

$$\dot{\alpha} = \mathrm{Im}\,\langle\tilde{\psi}|\dot{\tilde{\psi}}\rangle \tag{35}$$

$$= \frac{\partial}{\partial\epsilon}\langle\tilde{\psi}(t)|\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle\bigg|_{\epsilon=0} \tag{36}$$

$$= \frac{1}{2}\frac{\partial}{\partial\epsilon}\,\mathrm{Tr}\,P_{\psi}(t,\epsilon)\bigg|_{\epsilon=0} \tag{37}$$

(where $P_{\psi}(t,\epsilon) \equiv |\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle\langle\tilde{\psi}(t-\mathrm{i}\epsilon)|$). Imaginary time translation is effective in this case because of the close correspondence of equations (31) and (32)—that is, because imaginary unity is the generator of U(1) which is the holonomy group of $\mathcal{CP}^n$. Inhomogeneous coordinates are good coordinates on $\mathcal{CP}^n$ and in a coordinate representation the projective ket $|\tilde{\psi}(t)\rangle$, and its imaginary time-translated colleague $|\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle$ may be written as

$$|\tilde{\psi}(t)\rangle = \mathrm{e}^{K(t)}\begin{pmatrix}1\\ \vdots\\ w^n(t)\end{pmatrix} \to |\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle = \mathrm{e}^{K(t,\epsilon)}\begin{pmatrix}1\\ \vdots\\ w^n - \mathrm{i}\epsilon\dot{w}^n\end{pmatrix}. \tag{38}$$

The evaluation of the imaginary time-translated bracket $\langle\tilde{\psi}(t-\mathrm{i}\epsilon)|\tilde{\psi}(t-\mathrm{i}\epsilon)\rangle$ proceeds as follows:

$$\mathrm{Tr}\,P_{\psi}(t,\epsilon) = \mathrm{e}^{K(t,-\epsilon)}(1,\cdots,\bar{w}_n + \mathrm{i}\epsilon\dot{\bar{w}}_n)\mathrm{e}^{K(t,\epsilon)}\begin{pmatrix}1\\ \vdots\\ w^n - \mathrm{i}\epsilon\dot{w}^n\end{pmatrix} \tag{39}$$

$$= \mathrm{e}^{-(K(t,-\epsilon)+K(t,\epsilon))}\mathrm{e}^{2K(t,\epsilon)} \tag{40}$$

$$= \mathrm{e}^{K(t,\epsilon)-K(t,-\epsilon)} \tag{41}$$

whence it follows that

$$\frac{1}{2}\frac{\partial}{\partial\epsilon}\,\mathrm{Tr}\,P_{\psi}(t,\epsilon)\bigg|_{\epsilon=0} = \frac{\partial K}{\partial\epsilon}\bigg|_{\epsilon=0}. \tag{42}$$

In the following, we give some examples. Consider a particle in a spin state $|j, j\rangle$. As it is a pure spin state, its projective Hilbert space is a $\mathcal{CP}^1$ submanifold of $\mathcal{CP}^{2j}$ and as such is also Kähler. In terms of homogenous coordinates, it is given by [8]

$$Z^k = \cos^{2j}(\theta/2)\mathrm{e}^{-j\mathrm{i}\phi}\left(\frac{2j!}{(2j-k)!k!}\right)^{1/2}\tan^k(\theta/2)\mathrm{e}^{2k\mathrm{i}\phi} \tag{43}$$

i.e.

$$|j, j\rangle = \begin{pmatrix} \cos^{2j}(\theta/2)\mathrm{e}^{-j\mathrm{i}\phi} \\ (2j)^{\frac{1}{2}}\cos^{2j-1}(\theta/2)\sin(\theta/2)\mathrm{e}^{-(j-1)\mathrm{i}\phi} \\ \vdots \\ (2j)^{\frac{1}{2}}\cos(\theta/2)\sin^{2j-1}(\theta/2)\mathrm{e}^{(j-1)\mathrm{i}\phi} \\ \sin^{2j}(\theta/2)\mathrm{e}^{j\mathrm{i}\phi} \end{pmatrix}. \tag{44}$$

In order to transform to inhomogeneous coordinates, we may divide by Z^0 as this remains non-zero over the whole surface of the Riemann sphere, with the exception of the south pole. The inhomogeneous coordinates are therefore

$$w^k = \left(\frac{2j!}{(2j-k)!k!}\right)^{1/2}\tan^k(\theta/2)\mathrm{e}^{k\mathrm{i}\phi}. \tag{45}$$

If ϕ is identified with ωt, an imaginary time translation modifies w^k, giving

$$w^k(\epsilon) = \left(\frac{2j!}{(2j-k)!k!}\right)^{1/2}\tan^k(\theta/2)\mathrm{e}^{k\mathrm{i}\omega t}\mathrm{e}^{-k\omega\epsilon}. \tag{46}$$

This leads to a Kähler potential of

$$K(\epsilon) = \ln(1 + \tan^2(\theta/2)\mathrm{e}^{2\omega\epsilon})^j \tag{47}$$

and a phase one-form,

$$\Gamma = j\omega(1 - \cos\theta). \tag{48}$$

As a further example, consider the spin state $|j, 0\rangle$. This will be of the form

$$|j, 0\rangle = \begin{pmatrix} X_1(\theta)\mathrm{e}^{-j\mathrm{i}\phi} \\ X_2(\theta)\mathrm{e}^{-(j-1)\mathrm{i}\phi} \\ \vdots \\ X_{j+1} \\ \vdots \\ \pm X_2(\theta)\mathrm{e}^{(j-1)\mathrm{i}\phi} \\ \pm X_1(\theta)\mathrm{e}^{j\mathrm{i}\phi} \end{pmatrix} \tag{49}$$

where the symmetry in the functions X follows from the fact that under reflections we have $\theta \mapsto \pi - \theta$, $\phi \mapsto \pi + \phi$. The elements of the state vector transform as

$$|j, m\rangle_k \mapsto |j, -m\rangle_{-k} \tag{50}$$

where $-j \leqslant k \leqslant j$. This leads to a set of inhomogeneous coordinates:

$$\begin{pmatrix} 1 \\ \frac{X_2}{X_1}\mathrm{e}^{\mathrm{i}\phi} \\ \vdots \\ \frac{X_{j+1}}{X_1}\mathrm{e}^{j\mathrm{i}\phi} \\ \vdots \\ \frac{X_2}{X_1}\mathrm{e}^{(2j-1)\mathrm{i}\phi} \\ \mathrm{e}^{2j\mathrm{i}\phi} \end{pmatrix}. \tag{51}$$

After the customary substitutions, $\phi = \omega t, t \to t - \mathrm{i}\epsilon$, the Kähler potential becomes (after collecting terms up to $\mathrm{O}(\epsilon)$ only)

$$K = \tfrac{1}{2}\ln\left(2(1+2j\omega\epsilon) + 2\left(\frac{X_2}{X_1}\right)^2(1+2j\omega\epsilon) + \ldots + \left(\frac{X_{j+1}}{X_1}\right)^2(1+2j\omega\epsilon)\right)$$

$$= \tfrac{1}{2}\ln\left(2 + 2\left(\frac{X_2}{X_1}\right)^2 + \ldots + \left(\frac{X_{j+1}}{X_1}\right)^2\right) + \tfrac{1}{2}\ln(1+2j\omega\epsilon) \qquad (52)$$

$$= \tfrac{1}{2}\ln\left(2 + 2\left(\frac{X_2}{X_1}\right)^2 + \ldots + \left(\frac{X_{j+1}}{X_1}\right)^2\right) + j\omega\epsilon. \qquad (53)$$

To first order in ϵ,

$$K(t,\epsilon) = K(t,0) + \epsilon\Gamma \qquad (54)$$

so by comparison we see that for $|j,0\rangle$,

$$\Gamma = j\omega \qquad (55)$$

which implies

$$\gamma = 2\pi j = 0 \bmod 2\pi \qquad (56)$$

which arises from the decoupling of the θ and ϕ-dependencies.

In conclusion, in [1] the process of imaginary time translation was introduced as a means of obtaining the geometric phase. In this paper the scope of this method has been extended by application to Kähler manifolds but the principle remains the same in the equivalence of the mappings from Hilbert to projective Hilbert space and from homogeneous to inhomogeneous coordinates. The wider question of the reasons behind the efficacy of imaginary time translation as a mechanism has also been resolved.

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